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LETTER TO THE EDITOR

Exact fractals with adjustable fractal and fracton dimensionalities

D Ben-Avraham and S Havlin

Department of Physics, Bar-Ilan University, Ramat-Gan, Israel

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Abstract. A family of exact fractals is presented. By adjusting two external parameters, a wide range of fractal and fracton dimensionalities can be achieved. This includes the case of fracton dimensionality of 2, which is critical for diffusion.

Recently, there has been increasing interest in exact mathematical fractals (Mandelbrot 1977, 1982). Exact fractals have been shown to be a very useful model in describing percolating systems (Gefen *et al* 1981, Kirkpatrick 1979). The advantage of pure mathematical fractals is that one can generally calculate *exactly* different critical exponents related to their various properties. These exponents are easily related to analogous properties in percolation or in other systems which can be modelled by fractals. Much research is being carried out concerning the conductivity exponent μ and related problems, such as the anomalous diffusion on fractal-like spaces (Gefen *et al* 1981, 1983, Alexander and Orbach 1982, Ben-Avraham and Havlin 1982, Havlin *et al* 1983, Rammal and Toulouse 1983). Of particular interest is the question of the density of states of fractals (Alexander and Orbach 1982, Alexander 1983, Domany *et al* 1983) and its exponent, the fracton dimensionality (Alexander and Orbach 1982). In fact, the fracton dimensionality $\bar{\bar{d}}$ is related to diffusion in a special way. It seems that $\bar{\bar{d}}=2$ is a critical dimension in the sense that above this value, the exponent D for diffusion ($\langle R(t)\rangle^D \propto t$) is smaller than the fractal dimensionality $\bar{d}$, whereas below this value, D is greater than $\bar{d}$. This implies that $\bar{\bar{d}}=2$ plays the same role for anomalous diffusion as $d=2$ for normal diffusion. However, in spite of the interest in fractals with $\bar{\bar{d}}=2$, no such fractals appear in the literature (Rammal and Toulouse 1983).

In this letter, we present a family of exact mathematical fractals which are reminiscent of the Sierpinski sponge. All the exponents of interest can be easily calculated. By adjusting two external parameters, a wide range of values of $\bar{d}$ and $\bar{\bar{d}}$ can be achieved, including the interesting case of $\bar{\bar{d}}=2$ discussed above.

We define a family of exact fractals as follows. One starts with a d-dimensional hypercube which is subdivided to b^d smaller hypercubes. For $d=2$ the first stage of the fractal is a regular cartesian grid of $(b-bx)$ rows of b connected squares and bx rows each containing $(b-bx)$ squares. Each square belonging to the fractal is further diluted as in the first stage, and this procedure is continued indefinitely. In figure 1 we show an example of such a fractal embedded in a square with $b=5$ and $x=\frac{3}{5}$. Two stages of the procedure described above are shown. The figure is intended to show what we mean by 'a regular cartesian grid . . . '. Similarly for $d=3$ the first stage of the fractal is a regular cartesian grid built of $(b-bx)$ planes of cubes arranged as

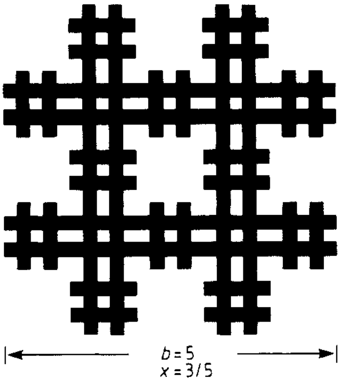

Figure 1. Example of a fractal embedded in a square lattice with $b=5$ and $x=\frac{3}{5}$. Two stages of the fractal are shown.

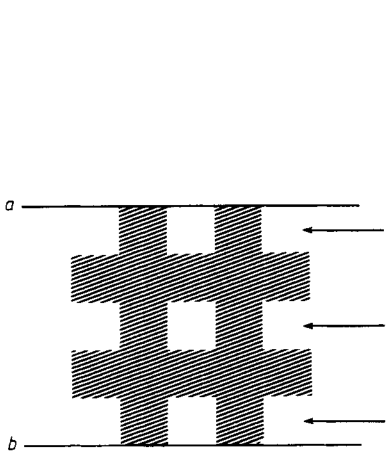

Figure 2. Calculation of the conductivity. A voltage is applied between the two lines a and b. The arrows point to the rows contributing to the first term in equation (2).

in the first stage of the fractal for $d=2$, which are connected by the remaining bx planes each consisting of $(b-bx)^2$ cubes. So for d dimensions the first stage of the fractal consists of $(b-bx)$ hyperplanes which are just built as the first stage of the $d-1$ fractal, and these planes are connected by bx hyperplanes each consisting of $(b-bx)^{d-1}$ hypercubes. Thus the fractal dimensionality $\bar{d}$ is generally given by

$$b^{\bar{d}} = bx(b-bx)^{d-1} + (b-bx)b^{\overline{d-1}}, \qquad (1)$$

where $b^{\bar{1}} = b$ by definition. This recursion can be solved to yield

$$\begin{aligned} b^{\bar{d}} &= b^d f_d(x) \equiv b^{d-\beta/\nu} \\ f_d(x) &= (1-x)^{d-1}[1+(d-1)x] = b^{-\beta/\nu}. \end{aligned} \qquad (2)$$

In order to derive the conductivity exponent μ/ν, define the exponent $\bar{\zeta}$ by $\rho(ba) = b^{\bar{\zeta}}\rho(a)$ and the relation $\mu/\nu = d-2+\bar{\zeta}$ (Gefen *et al* 1981). We cannot calculate the recursion formula for $\bar{\zeta}$ exactly, but in the following we shall present two different recursion formulae which give $\bar{\zeta}_1$ and $\bar{\zeta}_2$ such that $\bar{\zeta}_1 \leq \bar{\zeta} \leq \bar{\zeta}_2$, and we shall discuss a limit for which $\bar{\zeta}_1 \to \bar{\zeta}_2$. We first present the arguments for $d=2$. The whole fractal has a resistance $\rho(a)$. Upon magnification by b, each of the $b^2-(bx)^2$ squares comprising the fractal has the same original resistance $\rho(a)$. The resistance of the whole fractal is now $\rho(ba)$.

Imagine, for example, that one measures conductivity of a fractal between two lines as in figure 2. In the first approach, one ignores the internal structure of the squares in the succeeding stages and considers them as being 'full' and not further diluted. Clearly this underestimates the real resistance $\rho(ba)$, since the different squares are assumed to match all along their boundaries, which is not in fact correct. By this assumption,

$$b^{\bar{\zeta}_1} = xb/(b-xb) + (b-xb)/b = x/(1-x) + 1 - x, \qquad d=2. \qquad (3)$$

The first term represents the contribution of $(b-xb)$ parallel square resistances lying on xb different rows. These rows are marked with arrows in the example of figure 2. The second term represents the resistance of the remaining $(b-xb)$ rows which are in practice resistances with an area of b.

In the second approach, we neglect all the horizontal currents that might develop and thus only the contributions of $(b-bx)$ parallel chains of resistance $b\rho(a)$ contribute to $\rho(ba)$. Then

$$b^{\bar{\zeta}_2}=b/(b-xb)=1/(1-x), \qquad d=2. \tag{4}$$

This time we obviously overestimate the total resistance. Therefore, it follows that

$$\bar{\zeta}_1\leqslant\bar{\zeta}\leqslant\bar{\zeta}_2. \tag{5}$$

The above arguments can easily be extended for $d>2$. One has

$$b^{\bar{\zeta}_1}=xb/(b-xb)^{d-1}+(b-xb)/b^{\overline{d-1}}$$

or (6)

$$b^{d-2+\bar{\zeta}_1}=x/(1-x)^{d-1}+(1-x)/f_{d-1}(x)=b^{\mu_1/\nu}, \qquad d>2.$$

The first term represents the contribution of $(b-xb)^{d-1}$ parallel resistances lying on xb hyperplanes of dimension $d-1$. The second term takes into account the remaining $(b-xb)$ hyperplanes whose resistance is inversely proportional to their area b^{d-1}. The generalisation of the second approach is

$$b^{\bar{\zeta}_2}=b/(b-xb)^{d-1}$$

or (7)

$$b^{d-2+\bar{\zeta}_2}=1/(1-x)^{d-1}=b^{\mu_2/\nu}.$$

Now consider the case of $x=1-\varepsilon$ with $\varepsilon\ll 1$. Then, one obtains from equations (3) and (6) for any $d\geqslant 2$,

$$\mu_1/\nu=(1-d)\ln\varepsilon/\ln b-[(d-1)/d]\varepsilon/\ln b+O(\varepsilon^2) \tag{8}$$

and from equation (7),

$$\mu_2/\nu=(1-d)\ln\varepsilon/\ln b \qquad \text{(exact)}. \tag{9}$$

Since $\mu_1\leqslant\mu\leqslant\mu_2$ according to equation (5), one has

$$\mu/\nu=(1-d)\ln\varepsilon/\ln b-O(\varepsilon). \tag{10}$$

The fracton dimensionality $\bar{\bar{d}}$ is related to the exponents discussed above (Alexander and Orbach 1982) by

$$\bar{\bar{d}}=2\bar{d}/D \tag{11}$$

where D is the anomalous diffusion exponent (Ben-Avraham and Havlin 1982, Havlin *et al* 1983, Gefen *et al* 1983)

$$D=2+\mu/\nu-\beta/\nu=\bar{d}+\bar{\zeta}. \tag{12}$$

Thus, in fact, b and x determine $\bar{d}$ and $\bar{\bar{d}}$ of the fractal. In table 1, we present some numerical values for the exponents $\bar{d}$, $\bar{\bar{d}}$ and $\bar{\zeta}$ obtained for several values of b and x for $d=3$. For the exponents $\bar{\bar{d}}$ and $\bar{\zeta}$, the table gives two limiting values ($\bar{\zeta}$ and $\bar{\zeta}_2$) from (6) and (7). However, for $x=1-\varepsilon$, these two values coincide to $O(\varepsilon)$ and then only one value is displayed in table 1. It is clearly seen that a wide range of values of $\bar{d}$ and $\bar{\bar{d}}$ can be achieved. This includes the cases of $\bar{\bar{d}}$ greater than, equal to or smaller than 2. Moreover, it is seen from the table that more flexibility for $\bar{d}$ and $\bar{\bar{d}}$ is achieved

Table 1. Dependence of the exponents on b and x for $d=3$. Whenever there is a difference, the upper values refer to $\bar{\zeta}_1$ and the lower values to $\bar{\zeta}_2$.

x	b	$\bar{d}$	$\bar{\bar{d}}$	$\bar{\zeta}$
0.5	2	2.00	1.66	0.42
			1.33	1.00
	5	2.57	2.36	−0.39
			2.11	−0.14
0.8	5	1.59	1.29	0.88
			1.23	1.00
	20	2.24	1.99	0.01
			1.94	0.07
	50	2.42	2.21	−0.23
			2.16	−0.18
0.99	10^2	1.24	1.11	1.00
	10^4	2.12	2.00	0.00
	10^6	2.41	2.32	−0.33
0.999	10^3	1.16	1.07	1.00
	10^6	2.08	2.00	0.00
	10^9	2.39	2.32	−0.33

in the limit of $x=1-\varepsilon$ with $\varepsilon \ll 1$, for which one knows the exact solution to $O(\varepsilon)$. As a matter of fact,

$$\bar{\bar{d}} = 2\ln[b^d f_d(x)]/\ln[b^2(1-x)^{1-d} f_d(x)] + O(\varepsilon). \tag{13}$$

In conclusion, we have presented a fractal family with two adjustable parameters, which allow one to achieve a wide range of values $\bar{d}$ and $\bar{\bar{d}}$. It is interesting to note that from (1), (6) and (7), we can relate μ_1/β and μ_2/β to x and d independent of b, so one expects μ/β to have the same features. The limits $x\to 0$, $b=$ constant or $x=$ constant, $b\to\infty$ are equivalent, as seen from (1), (6) and (7). In this limit, the fractals behave like homogeneous space, that is, $d=\bar{d}=\bar{\bar{d}}$. It would be interesting to study possible models of percolation using these fractals or other similar adjustable fractals. It is clear that in this way, one is able to fix the desired exponents. However, we do not expect that the same numerical values of the parameters x and b will describe percolation in different dimensions.

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